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Packing Odd Circuits in Eulerian Graphs

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Let $\mathscr{C}$ be the clutter of odd circuits of a signed graph (G, Σ). For nonnegative integral edge-weights w, we are interested in the linear program $\min(w^t x\colon\ x(C) \geqslant 1$, for $C \in \mathscr{C}$, and $x \geqslant 0)$, which we denote by (P). The problem of solving the related integer program clearly contains the maximum cut problem, which is NP-hard. Guenin proved that (P) has an optimal solution that is integral so long as (G, Σ) does not contain a minor isomorphic to odd-K_5. We generalize this by showing that if (G, Σ) does not contain a minor isomorphic to odd-K_5 then (P) has an integral optimal solution and its dual has a half-integral optimal solution.

Key Words: strongly-bipartite; weakly bipartite; evenly bipartite; signed graphs.

1. INTRODUCTION

A *signed graph* is a pair (G, Σ) where G is an undirected graph and $\Sigma \subseteq E(G)$. We think of the edges in Σ as having odd length while the other edges have even length. A subset X of edges is called *odd* (resp. *even)* if $|X \cap \Sigma|$ is odd (resp. even). We denote the set of all odd circuits of (G, Σ) by $\mathscr{C}(G, \Sigma)$. The set $\mathscr{C}(G, \Sigma)$ is a *clutter*; that is, no element of $\mathscr{C}(G, \Sigma)$ properly contains another. We are interested in packings and coverings of this clutter. A subset $\mathscr{P}$ of $\mathscr{C}(G, \Sigma)$ is a *packing of odd circuits of* (G, Σ) if no two circuits in $\mathscr{P}$ share a common edge. A subset B of $E(G)$ is an *odd-circuit cover* of (G, Σ) if every odd circuit of (G, Σ) contains some edge of B. Evidently, if $\mathscr{P}$ is a packing of odd circuits and B is an odd-circuit cover then $|B| \geqslant |\mathscr{P}|$. Moreover, if $|B| = |\mathscr{P}|$ then B is a transversal of $\mathscr{P}$; that is, each circuit in $\mathscr{P}$ contains exactly one edge in B and each edge in B is contained in some circuit of $\mathscr{P}$. We say that (G, Σ) *packs* if there exists an odd-circuit cover B and a packing of odd circuits $\mathscr{P}$ with the same cardinality.

For $n \geqslant 3$, let odd-K_n denote the signed graph $(K_n, E(K_n))$, where K_n is the complete graph with n vertices. Note that odd-K_4 does not contain two edge

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disjoint odd circuits, but any odd-circuit cover of odd-K_4 has at least two edges. Therefore, odd-K_4 does not pack. Similarly, it can be checked that odd-K_5 does not pack.

There is a natural relationship between the problem of finding a minimum odd-circuit cover and the maximum cut problem. Let $G = (V, E)$ be a graph with edge-weights $w \in \mathbf{Z}_+^E$. The *maximum cut problem is* the problem of finding a subset U of V maximizing $w(\delta(U))$, where $\delta(U)$ denotes the set of edges having exactly one end in U, and $w(\delta(U)) := \Sigma(w_e\colon e \in \delta(U))$. Equivalently, one could look for a minimum weight set S of edges such that $G \backslash S$ is bipartite. (Where $G \backslash S$ is the graph obtained from G by deleting the edges in S.) That is, S is a minimum weight odd-circuit cover of (G, E).

Given edge-capacities $w \in \mathbf{Z}_+^{E(G)}$ consider the linear program

$$
\text{(P)} \quad \begin{cases} \text{Minimize} & \sum(w_e x_e\colon\ e \in E) \\ \text{subject to} & x(C) \geqslant 1,\ \ C \in \mathscr{C}(G, \Sigma), \\ & x_e \geqslant 0,\ \ e \in E, \end{cases}
$$

and its dual

$$
\text{(D)} \quad \begin{cases} \text{Maximize} & y(\mathscr{C}(G, \Sigma)) \\ \text{subject to} & \sum(y_C\colon\ e \in C \in \mathscr{C}(G, \Sigma)) \leqslant w_e,\ \ e \in E, \\ & y_C \geqslant 0,\ \ C \in \mathscr{C}(G, \Sigma). \end{cases}
$$

We say that (G, Σ) *packs with respect to* w if (P) and (D) both have optimal solutions that are integral. Define a signed graph (G', Σ') by replacing each edge e in (G, Σ) with w_e parallel edges. (Two edges are *parallel* in a signed graph if they have the same ends and the same sign.) Evidently, (G, Σ) packs with respect to w if and only if (G', Σ') packs. We say that (G, Σ) is *strongly bipartite* if (G, Σ) packs with respect to any nonnegative integral edge-capacities. Thus odd-K_4 is not strongly bipartite.

We call $\Sigma' \subseteq E(G)$ a *signature* of (G, Σ) if (G, Σ) and (G, Σ') have the same odd circuits. For example, if $U \subseteq V(G)$, then $\Sigma \Delta \delta(U)$ is a signature of (G, Σ). (Here Δ denotes symmetric difference.) It is straightforward to prove that $\Sigma' \subseteq E(G)$ is a signature of (G, Σ) if and only if there exists $U \subseteq V(G)$ such that $E' = \Sigma \Delta \delta(U)$. Obviously, any signature of (G, Σ) is an odd-circuit cover. While the converse is clearly not true, in general, it is straightforward to prove that any minimal odd-circuit cover is a signature.

If $S \subseteq E(G)$ then we let $(G, \Sigma) \backslash S$ denote the signed graph $(G \backslash S, \Sigma - S)$. If $S \subseteq E(G)$ and S is disjoint from Σ then we let $(G, \Sigma)/S$ denote the signed graph $(G/S, \Sigma)$, where G/S is the graph obtained from G by contracting the edges in S. More generally, if $S \subseteq E(G)$ and S does not contain an odd circuit, then there exists a signature Σ' that is disjoint from S, and we let $(G,\Sigma)/S$ denote $(G,\Sigma')/S$. While $(G,\Sigma)/S$ need not have a uniquely defined signature, its clutter of odd circuits is uniquely determined. We say that (G,Σ) is *isomorphic* to (G',Σ') if G is isomorphic to G' and up to relabeling edges of G', $\mathscr{C}(G,\Sigma)=\mathscr{C}(G',\Sigma')$. A signed graph (G',Σ') is a *minor* of (G,Σ) if there exist disjoint subsets X_d, X_c of $E(G)$ such that X_c does not contain an odd circuit and (G',Σ') is isomorphic to a graph obtained from $(G,\Sigma)\backslash X_d/X_c$ by possibly deleting isolated vertices. The family of strongly bipartite signed graphs is closed under taking minors, so one can characterize this family by describing the minor-minimal signed graphs not in the family.

1.1 (Seymour [10]). *A signed graph (G,Σ) is strongly bipartite if and only if (G,Σ) has no minor isomorphic to odd-K_4.*

Edge-capacities $w\in\mathbf{Z}_+^{E(G)}$ are called *Eulerian* if $w(\delta(v))$ is even for all $v\in V$. (Here $\delta(v)$ denotes $\delta(\{v\})$.) A signed graph (G,Σ) is called *evenly bipartite* if $\mathscr{C}(G,\Sigma)$ packs with respect to any Eulerian edge-capacities. While odd-K_4 is not strongly bipartite, it is straightforward to check that it is evenly bipartite. However, odd-K_5 is clearly not evenly bipartite. We prove the following conjecture of Bert Gerards (pers. comm.).

THEOREM 1.2. *A signed graph (G,Σ) is evenly bipartite if and only if (G,Σ) has no minor isomorphic to odd-K_5.*

Theorem 1.2 has a number of surprising corollaries, which we discuss now and in the next section. We call (G,Σ) *weakly bipartite* if (P) has an integral optimal solution for any nonnegative integral edge-capacities. Evidently, evenly bipartite signed graphs are also weakly bipartite. Indeed, suppose that (G,Σ) is evenly bipartite and $w\in\mathbf{Z}_+^{E(G)}$. Now, $2w$ is Eulerian, so there exists an integral optimal solution x to (P) with respect to the weights $2w$. Clearly, x is also optimal with respect to w. Thus, (G,Σ) is weakly bipartite, as claimed. The striking thing is that the converse also holds. In the light of Theorem 1.2, it suffices to show that odd-K_5 is not weakly bipartite. Consider odd-K_5 with unit edge-capacities. The smallest odd-circuit cover of odd-K_5 has 4 edges. However, assigning $x_e=\frac{1}{3}$ for all $e\in E(K_5)$ defines a feasible solution to (P) with objective value $\frac{10}{3}$. Therefore, odd-K_5 is not weakly bipartite. Thus we obtain the following theorem of Guenin [4] as a corollary.

COROLLARY 1.3. *A signed graph is weakly bipartite if and only if it does not contain a minor isomorphic to odd-K_5.*

Using the same trick, as above, of doubling the edge-capacities, we also obtain the following result.

COROLLARY 1.4. *Let* (G, Σ) *be an evenly bipartite signed graph. Then, for any edge-capacities* $w \in \mathbf{Z}_+^{E(G)}$, *the linear program* (P) *has an optimal solution that is integral and its dual* (D) *has an optimal solution that is half-integral.*

Schrijver [9] provided a very short proof of Corollary 1.3. Theorem 1.1 is actually a special case of a more general theorem of Seymour on binary clutters, and Guenin [5] provided a short proof of this more general theorem. Our proof of Theorem 1.2 combines ideas from these two proofs. We introduce some of these ideas by first proving Theorem 1.1.

2. MULTICOMMODITY FLOWS

In this section, we discuss the close connection between the packing and covering problems described in the introduction and multicommodity flows. We begin by defining the multi-commodity flow problem. We are given a signed graph (G, Σ), and a function $c \in \mathbf{Z}_+^{E(G)}$. An edge $d \in \Sigma$ is called a *demand edge*, and c_d is the *demand* on d. For $e \in E - \Sigma$, we call c_e the *capacity* of e. Let $\mathscr{C}_1$ be the set of all circuits C of G such that $|C \cap \Sigma| = 1$. Thus, if $C \in \mathscr{C}_1$ then there exists a demand edge $d \in \Sigma$ such that $C - \{d\}$ is a path connecting the ends of d. We say that $y \in \mathbf{R}_+^{\mathscr{C}_1}$ is a *fractional* (G, Σ, c)-*flow* if:

(1) for each $d \in \Sigma, \sum (y_P\colon\ d \in P \in \mathscr{C}_1) = c_d$, and
(2) for each $e \in E - \Sigma, \sum (y_P\colon\ e \in P \in \mathscr{C}_1) \leqslant c_e$.

The first condition requires that the demands are satisfied, and the second condition requires that the capacities are not exceeded. A natural condition for the existence of a fractional flow is that, the demand across a cut should not exceed its capacity. That is

2.1 (*Cut-condition*). *For all* $U \subset V$, $c(\delta(U) - \Sigma) \geqslant c(\delta(U) \cap \Sigma)$.

Gerards [2] notes that for a weakly bipartite signed graph, there exists a fractional flow if and only if the cut-condition is satisfied. Here, we are interested in integer flows. A flow y is an *integer flow* if $y \in \mathbf{Z}_+^{E(G)}$, and y is a *half-integer flow* if $2y \in \mathbf{Z}_+^{E(G)}$.

LEMMA 2.2. *Let* (G, Σ) *be a signed graph that packs with respect to edge-capacities* $c \in \mathbf{Z}_+^{E(G)}$. *Then, there exists an integer* (G, Σ, c)-*flow if and only if the cut-condition is satisfied.*

Proof. The cut-condition is clearly necessary for the existence of such a flow, so it suffices to prove the converse. Suppose that the cut-condition is satisfied. It follows that, for all $U \subseteq V(G)$,

$$c(\Sigma \Delta \delta(U)) \geqslant c(\Sigma).$$

Therefore, Σ is a minimum cost odd-circuit cover of (G, Σ). Thus, since (G, Σ) packs with respect to c, the characteristic vector of Σ is an optimal solution to (P). Let $y \in \mathbf{Z}_+^{\mathscr{C}(G,\Sigma)}$ be an optimal solution to (D). Now, by the complementary slackness conditions, we see that

(i) for $C \in \mathscr{C}(G, \Sigma)$, if $y_C > 0$ then $|C \cap \Sigma| = 1$, and
(ii) for each $d \in \Sigma, \sum (y_C\colon e \in C \in \mathscr{C}(G, \Sigma)) = c_d$.

Therefore, the restriction of y to $\mathscr{C}_1$ given an integer (G, Σ, c)-flow. ∎

Applying 2.2 to evenly bipartite signed graphs we obtain the following theorem.

THEOREM 2.3. *Let (G, Σ) be an evenly bipartite signed graph. Then, for any Eulerian edge-weights $c \in \mathbf{Z}_+^{E(G)}$, there exists an integer (G, Σ, c)-flow if and only if the cut-condition is satisfied.*

There is an analogous result for strongly bipartite graphs, which we choose to omit. Using the trick of doubling integer edge-weights, we obtain the following theorem.

THEOREM 2.4. *Let (G, Σ) be an evenly bipartite signed graph. Then, for any $c \in \mathbf{Z}_+^{E(G)}$, there exists a half-integer (G, Σ, c)-flow if and only if the cut-condition is satisfied.*

Theorems 2.2 and 2.3 have numerous applications. The following results give classes of evenly bipartite graphs; other classes are described by Gerards [2]. In each case, it is straightforward to verify that the signed graphs do not contain a minor isomorphic to odd-K_5 and, hence, that they are evenly bipartite.

2.5 (Hu [6] and Rothschild and Whinston [8]). *If (G, Σ) is a signed graph and $|\Sigma| = 2$, then (G, Σ) is evenly bipartite.*

2.6 (Seymour [11]). *If (G, Σ) is a signed graph and G is planar, then (G, Σ) is evenly bipartite.*

Gerards (pers. comm.) observed that the following signed graphs have no odd-K_5 minor, but, prior to proving Theorem 1.2, we did not know that they were evenly bipartite. Although, significant partial results of this ilk were obtained by Lomonosov [7].

2.7. *If (G, Σ) is a signed graph and Σ* is a *circuit of length* 5, *then (G, Σ) is evenly bipartite.*

2.8 (Gerards and Sebő [3]). *If (G, Σ)* is a signed graph that has an even face embedding on the Klein bottle then (G, Σ) *is evenly bipartite.*

3. STRONGLY BIPARTITE GRAPHS

In this section we prove Theorem 1.1, for which we require some additional definitions. If (G,Σ) is a signed graph and $S \subseteq E(G)$ then we denote by $(G,\Sigma)[S]$ the signed graph $(G[S], \Sigma \cap S)$, where $G[S]$ is the subgraph of G induced by S. That is $(G,\Sigma)[S]$ is obtained from $(G,\Sigma)\backslash(E(G) - S)$ by deleting all isolated vertices. We call a signed graph *bipartite* if it has no odd circuits. Finally, if x and y are vertices of a path P, then we denote by $P[x,y]$ the subpath of P with ends x and y.

Proof of Theorem 1.1. Let (G_0,Σ_0) be a minor-minimal-signed graph that is not strongly bipartite, and let $e_0 \in E(G_0)$. Now choose edge-capacities $w \in \mathbf{Z}_+^{E(G_0)}$ minimizing $w(E(G_0)) - \frac{3}{2}w_{e_0}$ such that the clutter of odd circuits of (G_0,Σ_0) does not pack with respect to w.

Let (G,Σ) be the signed graph obtained from (G_0,Σ_0) by replacing each edge $f \in E(G_0)$ with w_f parallel edges. Thus, the clutter of odd circuits of (G,Σ) does not pack. Let e be one of the copies of e_0 in G, and let x and y be the ends of e.

Choose a set $\mathscr{C}$ of odd circuits of (G,Σ) such that:

(i) The sets $\{C - \{e\}\colon C \in \mathscr{C}\}$ are pairwise disjoint.
(ii) $\mathscr{C}$ has maximum cardinality with respect to (i).
(iii) $\{C\colon C \in \mathscr{C},\ e \in C\}$ has minimum cardinality with respect to (i) and (ii).
(iv) $\sum(|C|\colon C \in \mathscr{C})$ is minimum with respect to (i)–(iii).

Now let $(\mathscr{C}_e, \mathscr{C}_{\bar{e}})$ be the partition of $\mathscr{C}$ into circuits containing e and circuits not containing e, respectively.

3.1. $|\mathscr{C}_e| = 2$.

Proof. Let $\mathscr{C}' = \{C - \{e\}\colon C \in \mathscr{C}\}$. Then $\mathscr{C}'$ is a maximum packing of odd circuits in $(G,\Sigma)/e$. Now $(G,\Sigma)/e$ packs, so there exists an odd-circuit cover B of $(G,\Sigma)/e$ such that $|B| = |\mathscr{C}'| = |\mathscr{C}|$. Evidently, B is also an odd-circuit cover of (G,Σ). However, (G,Σ) does not pack, so $|\mathscr{C}_e| \geqslant 2$. Suppose that $|\mathscr{C}|_e > 2$.

Construct a signed graph (G_1,Σ_1) by adding to (G,Σ) an edge e_1 parallel with e. By our choice of w, we see that (G_1,Σ_1) packs. Let $\mathscr{C}_1$ be a maximum cardinality packing of odd circuits in (G_1,Σ_1), and let B_1 be a minimum odd-circuit cover of (G_1,Σ_1). Thus, $|\mathscr{C}_1| = |B_1|$. By our choice of $\mathscr{C}$, and since $|\mathscr{C}_e| > 2$, we must have $|\mathscr{C}_1| < |\mathscr{C}|$. Therefore, $|B_1| < |\mathscr{C}|$. Nevertheless, B_1 must intersect each of the odd circuits in $\mathscr{C}$, so we must have $e \in B_1$. Moreover, since e_1 is in parallel with e, we also have $e_1 \in B_1$.

Now, $B_1 - \{e_1\}$ is an odd-circuit cover of (G,Σ), and there exists a packing $\mathscr{C}' \subset \mathscr{C}_1$ of odd circuits of (G,Σ) such that $|\mathscr{C}'| = |\mathscr{C}_1| - 1 = |B_1 - \{e_1\}|$, contrary to the fact that (G,Σ) does not pack. ∎

Let C_1 and C_2 be the two odd circuits in $\mathscr{C}_e$.

3.2. *For any odd circuit $C \subseteq C_1 \cup C_2$ of (G, Σ) there exists an odd-circuit cover B of (G, Σ) such that $B - C$ is a transversal of $\mathscr{C}_{\bar{e}}$.*

Proof. By the definition of $\mathscr{C}$, $\mathscr{C}_{\bar{e}}$ is a maximum packing of odd circuits of $(G, \Sigma)\backslash C$. However, by our choice of w, $(G, \Sigma)\backslash C$ packs. Therefore, there exists an odd-circuit cover B' of $(G, \Sigma)\backslash C$ that is a transversal of $\mathscr{C}_{\bar{e}}$. Now B' extends to an odd-circuit cover B of (G, Σ) with the desired properties. ∎

Let $P_1 := C_1 - \{e\}$ and $P_2 := C_2 - \{e\}$. Thus P_1 and P_2 are both (x, y)-paths. These paths are edge disjoint but not necessarily internally vertex disjoint. Nevertheless, P_1 and P_2 do not intersect wildly.

3.3. *$(G, \Sigma)[P_1 \cup P_2]$ is bipartite and if v is a common vertex of P_1 and P_2 then $P_1[x, v] \cup P_2[v, y]$ and $P_2[x, v] \cup P_1[v, y]$ are both (x, y)-paths.*

Proof. Suppose that $(G, \Sigma)[P_1 \cup P_2]$ contains an odd circuit C'. Then, there exists an odd circuit $C \subseteq C_1 \Delta C_2 \Delta C'$ in (G, Σ). Now $C_{\bar{e}} \cup \{C, C'\}$ contradicts (iii) of our choice of $\mathscr{C}$. Thus $(G, \Sigma)[P_1 \cup P_2]$ is indeed bipartite.

Now $P_1[x, v] \cup P_2[v, y]$ and $P_2[x, v] \cup P_1[v, y]$ contain (x, y)-paths, say P_1' and P_2', respectively. Since $(G, \Sigma)[P_1 \cup P_2]$ is bipartite, $P_1' \cup \{e\}$ and $P_2' \cup \{e\}$ are both odd circuits. By our choice (iv) of $\mathscr{C}$ we must have $P_1' \cup P_2' \cup \{e\} = C_1 \cup C_2$. Therefore, $P_1' = P_1[x, v] \cup P_2[v, y]$ and $P_2' = P_2[x, v] \cup P_1[v, y]$, as required. ∎

The following technical claim allows us to fully disentangle P_1 and P_2.

3.4. *There exists a minor (G', Σ') of (G, Σ) and odd circuits $C_1', C_2' \subseteq C_1 \cup C_2$ of (G', Σ') such that*

(i) *$E(G) - E(G') \subseteq P_1 \cup P_2$.*
(ii) *$C_1' - \{e\}$ and $C_2' - \{e\}$ are internally vertex disjoint (x, y)-paths in G'.*
(iii) *For $i \in \{1, 2\}$, there exists an odd-circuit cover B' of (G', Σ') such that $B' - C_i'$ is a transversal of $\mathscr{C}_{\bar{e}}$.*
(iv) *For any transversal T of $\mathscr{C}_{\bar{e}}$, $T \cup \{e\}$ is not an odd-circuit cover of (G', Σ').*

Proof. Choose a minor (G', Σ') of (G, Σ) that is minimal subject to:

(1) $E(G) - E(G') \subseteq P_1 \cup P_2$.
(2) There exist odd circuits $C_1', C_2' \subseteq C_1 \cup C_2$ of (G', Σ') such that $C_1' \cap C_2' = \{e\}$.
(3) For any odd circuit $C \subseteq C_1 \cup C_2$ of (G', Σ') there exists an odd-circuit cover B' of (G', Σ') such that $B' - C$ is a transversal of $\mathscr{C}_{\bar{e}}$.

(4) For any transversal T of $\mathscr{C}_{\bar{e}}$, $T \cup \{e\}$, is not an odd-circuit cover of (G', Σ').

Note that these conditions are satisfied by (G, Σ), so (G', Σ') is well defined. Let C_1' and C_2' be the circuits of G' given in (2). Now, (G', Σ'), C_1' and C_2' satisfy (i), (iii), and (iv). So it remains to prove that (ii) is satisfied. Let $P_1' := C_1' - \{e\}$ and $P_2' := C_2' - \{e\}$. Suppose that P_1' and P_2' are not internally vertex disjoint and let v be a common internal vertex. We break the proof into two cases.

Case 1. There exists an (x, v)-path $P \subseteq P_1'[x, v] \cup P_2'[x, v]$ such that there does not exist any odd-circuit cover B of (G', Σ') such that $B - (P \cup \{e\})$ is a transversal of $\mathscr{C}_{\bar{e}}$.

Let P' be an (x, v)-path in $P \Delta P_1'[x, v] \Delta P_2'[x, v]$. Now $(G', \Sigma') \backslash P / P'$ certainly satisfies (1) and (2). Moreover, by our choice of P, it is straightforward to check that $(G', \Sigma') \backslash P / P'$ satisfies (4). Now, if $C \subseteq C_1 \cup C_2$ is an odd circuit of $(G', \Sigma') \backslash P / P'$ then, by 3.3, $P \cup C$ is an odd circuit in (G', Σ'). Then, since (G', Σ') satisfies (3) we easily deduce that $(G', \Sigma') \backslash P / P'$ also satisfies (3). However, this contradicts the minimality of (G', Σ').

Case 2. For any (x, v)-path $P \subseteq P_1'[x, v] \cup P_2'[x, v]$ there exists an odd-circuit cover B of (G', Σ') such that $B - (P \cup \{e\})$ is a transversal of $\mathscr{C}_{\bar{e}}$.

Let $(G'', \Sigma'') := (G', \Sigma') / (P_1'[v, y] \cup P_2'[v, y])$. Now (G'', Σ'') certainly satisfies (1), (2) and (4). Moreover, by the hypothesis of this case, it is easy to see that (G'', Σ'') also satisfies (3). However, this contradicts the minimality of (G', Σ').

Let (G', Σ'), C_1' and C_2' be as given in 3.4, and, for $i \in \{1, 2\}$, let B_i' be a minimal odd-circuit cover of (G', Σ') such that $B_i' - C_i'$ is a transversal of $\mathscr{C}_{\bar{e}}$. As B_1' and B_2' are both minimal odd-circuit covers, they are also both signatures of (G', Σ'), and, hence, $B_1' \Delta B_2'$ is a cut. Moreover, $e \notin B_1' \Delta B_2'$. Therefore, there exists $U \subseteq V(G')$ such that $\delta_{G'}(U) = B_1' \Delta B_2'$ and $x, y \notin U$. Let $X_1 := V(C_1') \cap U$ and let $X_2 := V(C_2') \cap U$.

3.5. *There exists a path from X_1 to X_2 in $G'[U] \backslash B_1'$.*

Proof. We first show that there exists a path from X_1 to X_2 in $G'[U]$. Suppose not, then there exists $X \subseteq U$ such that $X_1 \subseteq X$, $X \subseteq U - X_2$, and $\delta_{G'[U]}(X) = \emptyset$. Let $B := B_1 \Delta \delta_{G'}(X)$. Now, B is an odd-circuit cover of (G', Σ'), and $B - \{e\}$ is a transversal of $\mathscr{C}_{\bar{e}}$, contrary to part (iv) of 3.4. Thus, there exists a path from X_1 to X_2 in $G'[U]$ as claimed.

Let P be a path from X_1 to X_2 in $G'[U]$ minimizing $|P \cap B_1'|$. Suppose that there exists an edge $f \in P \cap B_1'$. Since $f \notin \delta_{G'}(U)$, we see that $f \in B_2'$. Thus, there exists an odd circuit $C \in \mathscr{C}_{\bar{e}}$ of (G', Σ') containing f, and, by the definition of B_1', we have $C \cap B_1' = \{f\}$. Therefore, C is a circuit of $G'[U]$. We can obtain another path P' from X_1 to X_2 in $G'[U]$ by rerouting P

through $C - \{f\}$. However $|P' \cap B_1'| < |P \cap B_1'|$, which contradicts our choice of P. We conclude that P is in fact disjoint from B_1'. ∎

Let P be a path from X_1 to X_2 in $G'[U] \backslash B_1'$. Now, B_1' is a signature for (G', Σ'), and it is straightforward to check that $(G', B_1')[C_1' \cup C_2' \cup P]$ is a subdivision of odd-K_4.

4. EVENLY BIPARTITE GRAPHS

In this section, we prove Theorem 1.2. The proof is very similar to that of Theorem 1.1, but to obtain the odd-K_5-minor at the end, we require the following lemma. (This lemma is essentially due to Schrijver [9].)

LEMMA 4.1. *Let $G = (V, E)$ be a graph, let e be an edge of G with ends x and y, let (Y_0, Y_1, Y_2, Y_3) be disjoint subsets of V, and let P_1, P_2, and P_3 be internally vertex disjoint (x, y)-paths in $G \backslash e$. Moreover, suppose that*

(1) *$x, y \in Y_0$ and, for $i \in \{0, 1, 2, 3\}$, Y_i is a stable set of $G \backslash e$,*
(2) *for $i \in \{1, 2, 3\}$, $V(P_i) \subseteq Y_0 \cup Y_i$, and*
(3) *for distinct $i, j \in \{1, 2, 3\}$, there exists a path from $V(P_i)$ to $V(P_j)$ in $G[Y_i \cup Y_j]$.*

Then $(G, E(G))$ has a minor isomorphic to odd-K_5.

Proof. Suppose otherwise, and let G be a counterexample minimizing $|V(G)| + |E(G)|$. For distinct $i, j \in \{1, 2, 3\}$, let P_{ij} be a path from $V(P_i)$ to $V(P_j)$ in $G[Y_i \cup Y_j]$. (We assume that $P_{ij} = P_{ji}$.) By the minimality of G, we have $E(G) := \{e\} \cup P_1 \cup P_2 \cup P_3 \cup P_{12} \cup P_{23} \cup P_{13}$, and $V(G) := V(P_1) \cup V(P_2) \cup V(P_3) \cup V(P_{12}) \cup V(P_{23}) \cup V(P_{13})$.

Suppose that G has a vertex v of degree 2, and define $G' := G / \delta_G(v)$. Note that $(G, E(G)) / \delta_G(v) = (G', E(G'))$, and that G' satisfies the conditions of the lemma. However, this contradicts the minimality of G, and, hence, G has no vertices of degree 2. Thus, we see that $Y_0 = \{x, y\}$, and, for each $i \in \{1, 2, 3\}$, P_i has exactly one internal vertex, say v_i.

Now, the neighbours of x are v_1, v_2, v_3, and y, and the neighbours of y are v_1, v_2, v_3, and x. Moreover, since G has no vertices of degree 2, we also conclude that $Y_1 = V(P_{12}) \cap V(P_{13})$, $Y_2 = V(P_{12}) \cap V(P_{23})$, and $Y_3 = (P_{13}) \cap V(P_{23})$. Therefore, $|Y_1| = |Y_2| = |Y_3|$.

If $|Y_1| = 1$, then $(G, E(G))$ is isomorphic to odd-K_5, so we may assume that $|Y_1| > 1$. For distinct $i, j \in \{1, 2, 3\}$, let e_{ij} be the edge on P_{ij} that is incident with v_i. Let $G' := G \backslash \{e_{13}, e_{32}, e_{21}\} / e_{12}, e_{23}, e_{31}\}$, and, for distinct $i, j \in \{1, 2, 3$ $\}$, let $P_{ij}' := P_{ij} - \{e_{ij}, e_{ji}\}$. Now let $Y_1' := V(P_{12}') \cap V(P_{13}')$, $Y_2' := V(P_{12}') \cap V(P_{23}')$, $Y_3' := V(P_{13}') \cap V(P_{23}')$, and $Y_0' := \{x, y\}$. Note that $(G', E(G'))$ is a minor of $(G, E(G))$ and that G' satisfies the conditions of the lemma. However, this contradicts the minimality of G. ∎

Proof of Theorem 1.2. Let (G_0, Σ_0) be a signed graph that is minor-minimally and not evenly bipartite, and let $e_0 \in E(G_0)$. Now choose Eulerian edge-capacities $w \in \mathbf{Z}_+^{E(G_0)}$ minimizing $w(E(G_0)) - \frac{3}{2}w_{e_0}$ such that the clutter of odd circuits of (G_0, Σ_0) does not pack with respect to w.

Let (G, Σ) be the signed graph obtained from (G_0, Σ_0) by replacing each edge $f \in E(G_0)$ with w_f parallel edges. Thus, the clutter of odd circuits of (G, Σ) does not pack. Note that G is Eulerian; that is, all its vertices have even degree. Let e be one of the copies of e_0 in G, and let x and y be the ends of e.

Choose a set $\mathscr{C}$ of odd circuits of (G, Σ) such that:

(i) The sets $(C - \{e\}\colon C \in \mathscr{C})$ are pairwise disjoint.
(ii) $\mathscr{C}$ has maximum cardinality with respect to (i).
(iii) $\{C\colon C \in \mathscr{C},\ e \in C\}$ has minimum cardinality with respect to (i) and (ii).
(iv) $\sum(|C|\colon C \in \mathscr{C})$ is a minimum with respect to (i)–(iii).

Now let $(\mathscr{C}_e, \mathscr{C}_{\bar{e}})$ be the partition of $\mathscr{C}$ into circuits containing e and circuits not containing e, respectively.

4.2. $|\mathscr{C}_e| = 3$.

Proof. Let $\mathscr{C}' = \{C - \{e\}\colon C \in \mathscr{C}\}$. Then $\mathscr{C}'$ is a maximum packing of odd circuits in $(G, \Sigma)/e$. Now G/e is Eulerian, so, by the minimality of (G, Σ), $(G, \Sigma)/e$ packs. Thus, there exists an odd-circuit cover B of $(G, \Sigma)/e$ such that $|B| = |\mathscr{C}'| = |\mathscr{C}|$. Evidently, B is also an odd-circuit cover of (G, Σ). However, (G, Σ) does not pack, so $|\mathscr{C}_e| \geqslant 2$.

Now, we claim that $|\mathscr{C}_e|$ is odd. Suppose otherwise. Let $S = \bigcup(C\colon C \in \mathscr{C})$, and consider $(G, \Sigma)\backslash S$. Note that x and y are the only two vertices in $G\backslash S$ with odd degree. Therefore, there exists an (x, y)-path P in $G\backslash S$. If $P \cup \{e\}$ is an odd circuit of (G, Σ) then $\mathscr{C} \cup \{P \cup \{e\}\}$ contradicts our choice (ii) of $\mathscr{C}$. Thus $P \cup \{e\}$ is even. Let $C \in \mathscr{C}_e$. Then, $C\Delta(P \cup \{e\})$ contains an odd circuit C' of (G, Σ). Now, since $e \notin C'$, we see that $(\mathscr{C} - \{C\}) \cup \{C'\}$ contradicts our choice (iii) of $\mathscr{C}$. We conclude that $\mathscr{C}_e$ is odd, as claimed. Thus $|\mathscr{C}_e| \geqslant 3$. Suppose that $|\mathscr{C}_e| > 3$.

Construct a signed graph (G_1, Σ_1) by adding two edges e_1 and e_2 in parallel with e. Thus G_1 is Eulerian, and, by our choice of w, we see that (G_1, Σ_1) packs. Let $\mathscr{C}_1$ be a maximum cardinality packing of odd circuits in (G_1, Σ_1), and let B_1 be a minimum odd-circuit cover of (G_1, Σ_1). Thus, $|\mathscr{C}_1| = |B_1|$. By our choice of $\mathscr{C}$, and since $|\mathscr{C}_e| > 3$, we must have $|\mathscr{C}_1| < |\mathscr{C}|$. Therefore, $|B_1| < |\mathscr{C}|$. Nevertheless, B_1 must intersect each of the odd circuits in $\mathscr{C}$, so we must have $e \in B_1$. Moreover, since e_1 and e_2 are in parallel with e, we also have $e_1, e_2 \in B_1$.

Now, $B_1 - \{e_1, e_2\}$ is an odd-circuit cover of (G, Σ), and there exists a packing $\mathscr{C}' \subset \mathscr{C}_1$ of odd circuits of (G, Σ) such that $|\mathscr{C}'| = |\mathscr{C}_1| - 2 = |B_1 - \{e_1, e_2\}|$, contrary to the fact that (G, Σ) does not pack. ∎

Let C_1, C_2, and C_3 be the three odd circuits in $\mathscr{C}_e$, and let $P_i := C_i - \{e\}$ for $i \in \{1, 2, 3\}$. Thus P_1, P_2, and P_3 are (x, y)-paths. These paths are edge disjoint but not necessarily internally vertex disjoint. We first show that no two of these paths intersect wildly.

4.3. *For distinct* $i, j \in \{1, 2, 3\}$, $(G, \Sigma)[P_i \cup P_j]$ *is bipartite and if* v *is a common vertex of* P_i *and* P_j *then* $P_i[x, v] \cup P_j[v, y]$ *and* $P_j[x, v] \cup P_i[v, y]$ *are both* (x, y)*-paths.*

Proof. By symmetry we may assume that $i = 1$ and $j = 2$. Suppose that $(G, \Sigma)[P_1 \cup P_2]$ contains an odd circuit C'. Then, there exists an odd circuit $C \subseteq C_1 \Delta C_2 \Delta C'$ in (G, Σ). Now $\mathscr{C}_{\bar{e}} \cup \{C, C', C_3\}$ contradicts (iii) of our choice of $\mathscr{C}$. Thus $(G, \Sigma)[P_1 \cup P_2]$ is indeed bipartite.

Now $P_1[x, v] \cup P_2[v, y]$ and $P_2[x, v] \cup P_1[v, y]$ contain (x, y)-paths, say P_1' and P_2', respectively. Since $(G, \Sigma)[P_1 \cup P_2]$ is bipartite, $P_1' \cup \{e\}$ and $P_2' \cup \{e\}$ are both odd circuits. By our choice of $\mathscr{C}$ we must have $P_1' \cup P_2' \cup \{e\} = C_1 \cup C_2$. Therefore, $P_1' = P_1[x, v] \cup P_2[v, y]$ and $P_2' = P_2[x, v] \cup P_1[v, y]$, as required. ∎

4.4. $(G, \Sigma)[P_1 \cup P_2 \cup P_3]$ *is bipartite.*

Proof. We may assume $e \in \Sigma$. For $i \in \{1, 2, 3\}$, since C_i is an odd circuit and $e \in \Sigma$, there exists a unique subset U_i of $V(P_i)$ such that $x \in U_i$, $y \notin U_i$ and $P_i \cap \delta(U_i) = P_i \cap \Sigma$. By 4.3, for distinct $i, j \in \{1, 2, 3\}$, we see that $(P_i \cup P_j) \cap \delta(U_i \cup U_j) = (P_i \cup P_j) \cap \Sigma$. It follows that $(P_1 \cup P_2 \cup P_3) \cap \delta(U_1 \cup U_2 \cup U_3) = (P_1 \cup P_2 \cup P_3) \cap \Sigma$. Thus, $(G, \Sigma)[P_1 \cup P_2 \cup P_3]$ is bipartite, as claimed. ∎

Let H be the directed graph obtained by directing the edges of $G[P_1 \cup P_2 \cup P_3]$ such that P_1, P_2, and P_3 are (x, y)-dipaths.

4.5. *H is acyclic, and, for any (x, y)-dipath P in H, $P \cup \{e\}$ is an odd circuit of (G, Σ).*

Proof. Since $(G, \Sigma)[P_1 \cup P_2 \cup P_3]$ is bipartite, $P \cup \{e\}$ is certainly an odd circuit of (G, Σ). Now, define $f_a := 1$ for all $a \in E(H)$. Then, by construction, f is an (x, y)-flow in H of value 3. Now suppose that C is a directed circuit in H. Change f by setting $f_a := 0$ for all $a \in C$. Thus, f is still an (x, y)-flow in H of value 3. Therefore, there exists 3 edge-disjoint (x, y)-dipaths P_1', P_2', and P_3' in $H \backslash C$. Now, $P_1' \cup \{e\}$, $P_2' \cup \{e\}$, and $P_3' \cup \{e\}$ are all odd circuits of (G, Σ). Replacing C_1, C_2 and C_3 with these circuits gives a contradiction to condition (iv) of our choice of $\mathscr{C}$. Therefore, H is acyclic, as claimed. ∎

4.6. *If P is an (x, y)-dipath in H there exists two other (x, y)-dipaths P_1' and P_2' such that P, P_1' and P_2' are edge disjoint.*

Proof. This follows easily from the fact that H is acyclic. ∎

4.7. *If P is an (x, y)-dipath in H then there exists an odd-circuit cover B of (G, Σ) such that $B - (P \cup \{e\})$ is a transversal of $\mathscr{C}_{\bar{e}}$.*

Proof. By 4.6, there exist (x, y)-dipaths P_1' and P_2' in H such that P, P_1', and P_2' are edge-disjoint. Let $C_1' := P_1' \cup \{e\}$, $C_2' := P_2' \cup \{e\}$ and $C_3' := P \cup \{e\}$. Now, by our choice of w, $(G, \Sigma)\backslash C_3'$ packs. Let B' be a minimum odd-circuit cover of $(G, \Sigma)\backslash C_3'$, and let $B \subseteq B' \cup C_3'$ be a minimal odd-circuit cover of (G, Σ). If $\mathscr{C}_{\bar{e}}$ is a maximum packing of odd circuits of $(G, \Sigma)\backslash C_3'$ then B' is a transversal of $\mathscr{C}_{\bar{e}}$, and, hence, B has the desired properties. Therefore, we may assume that $\mathscr{C}_{\bar{e}}$ is not a maximum packing of odd circuits of $(G, \Sigma)\backslash C_3'$. Nevertheless, by our choice of $\mathscr{C}$, there is no packing of odd circuits of $(G, \Sigma)\backslash C_3'$ of size $|\mathscr{C}| - 1 = |\mathscr{C}_{\bar{e}}| + 2$. Therefore, $|B'| = |\mathscr{C}_{\bar{e}}| + 1$.

Since B is a minimal odd-circuit cover, B is a signature for (G, Σ). Therefore, B has an odd intersection with each circuit in $\mathscr{C}_{\bar{e}} \cup \{C_1', C_2', C_3'\}$. However, $|B'| = |\mathscr{C}_{\bar{e}}| + 1$. Therefore, there exists a unique edge $f \in B'$ that is not contained in a circuit in $\mathscr{C}_{\bar{e}}$. Now f is not contained in both C_1' and C_2', so we must have $e \in B$. Moreover, as C_1' and C_2' both have an odd intersection with B, neither C_1' nor C_2' contains f. We conclude that f is in none of the circuits in $\mathscr{C}_{\bar{e}} \cup \{C_1', C_2', C_3'\}$. Let G' be the graph obtained from G by deleting all of the edges in the circuits in $\mathscr{C}_{\bar{e}} \cup \{C_1', C_2', C_3'\}$. Note that G' is Eulerian. Therefore, there is a circuit C of G' that contains f. However, $|C \cap B| = 1$, so C is an odd circuit of (G, Σ). Then, $\mathscr{C}_{\bar{e}} \cup \{C, C_1', C_2', C_3'\}$ contradicts our choice of $\mathscr{C}$. ∎

The following technical claim allows us to fully disentangle P_1, P_2, and P_3.

4.8. *There exists a minor (G', Σ') of (G, Σ) and odd circuits $C_1', C_2', C_3' \subseteq C_1 \cup C_2 \cup C_3$ of (G', Σ') such that*:

- (i) $E(G) - E(G') \subseteq P_1 \cup P_2 \cup P_3$,
- (ii) $C_1' - \{e\}$, $C_2' - \{e\}$, *and* $C_3' - \{e\}$ *are internally vertex disjoint* (x, y)-*paths in* G'.
- (iii) *for* $i \in \{1, 2, 3\}$, *there exists an odd-circuit cover* B' *of* (G', Σ') *such that* $B' - C_i'$ *is a transversal of* $\mathscr{C}_{\bar{e}}$.
- (iv) *for any transversal* T *of* $\mathscr{C}_{\bar{e}}$, $T \cup \{e\}$ *is not an odd-circuit cover of* (G', Σ').

Proof. Let (G', Σ') be a minor of (G, Σ) and let H' be a directed graph obtained by orienting edges in a subgraph of G', where (G', Σ') and H' are minimal subject to:

(1) $E(G) - E(G') \subseteq P_1 \cup P_2 \cup P_3$, and $E(H') \subseteq P_1 \cup P_2 \cup P_3$.
(2) H' is acyclic and there exist three edge disjoint (x, y)-dipaths in H'.
(3) For any (x, y)-dipath P of H', $P \cup \{e\}$ is an odd circuit, and there exists an odd-circuit cover B' of (G', Σ') such that $B' - (P \cup \{e\})$ is a transversal of $\mathscr{C}_{\bar{e}}$.
(4) For any transversal $T$of $\mathscr{C}_{\bar{e}}$, $T \cup \{e\}$ is not an odd-circuit cover of (G', Σ').

Note that these conditions are satisfied by (G, Σ) and H, so (G', Σ') and H' are well defined. Let P_1', P_2' and P_3' be edge-disjoint (x, y)-dipaths in H', and let $C_i' = P_i' \cup \{e\}$ for $i \in \{1, 2, 3\}$. By the minimality of H', we have $E(H') = P_1' \cup P_2' \cup P_3'$. Now, (G', Σ'), C_1', C_2', and C_3' satisfy (i), (iii), and (iv), so it remains to prove that (ii) is satisfied. Suppose that P_1', P_2', and P_3' are not internally vertex disjoint. Let $v_1 \in V(P_1') - \{x\}$ be the closest vertex on P_1' to x that lies on P_2' or P_3'. Define v_2 and v_3 similarly. Since H' is acyclic it must be the case that at least two of v_1, v_2 and v_3 coincide. By symmetry we may assume that $v_1 = v_2$.

4.8.1. *For each* $i \in \{1, 2\}$, *there exists an odd-circuit cover* B *of* (G', Σ') *such that* $B - (P_i'[x, v_i] \cup \{e\})$ *is a transversal of* $\mathscr{C}_{\bar{e}}$.

Suppose otherwise. By symmetry, we may assume that there does not exist an odd-circuit cover B of (G', Σ') such that $B - (P_1'[x, v_1] \cup \{e\})$ is a transversal of $\mathscr{C}_{\bar{e}}$. Let $(G'', \Sigma'') := (G', \Sigma') \backslash P_1'[x, v_1] / P_2'[x, v_1]$. If $v_1 \neq v_3$ then let $H'' := H' \backslash P_1'[x, v_1] / P_2'[x, v_1]$, otherwise, when $v_1 = v_3$, let $H'' := H' \backslash P_1'[x, v_1] / (P_2'[x, v_1] \cup P_3'[x, v_1])$. Now (G'', Σ'') and H'' certainly satisfy (1), and (2). Moreover, it is straightforward to check that (G'', Σ'') satisfies (4). Now, if P is an (x, y)-dipath of H'' then $P_1'[x, v_1] \cup P$ is an (x, y)-dipath in H'. Then, since (G', Σ') and H' satisfy (3), we easily deduce that (G'', Σ'') and H'' also satisfy (3). However, this contradicts the minimality of (G', Σ') and H'. This proves 4.8.1.

4.8.2. *There exists a* (v_3, y)*-dipath* $\tilde{P}_3$ *such that, for each odd-circuit cover* B *of* (G', Σ'), $B - (\tilde{P}_3 \cup \{e\})$ *is not a transversal of* $\mathscr{C}_{\bar{e}}$.

Suppose otherwise. Thus, for any (v_3, y)-dipath P in H' there exists an odd-circuit cover B of (G', Σ') such that $B - (P \cup \{e\})$ is a transversal of $\mathscr{C}_{\bar{e}}$. Let $(G'', \Sigma'') := (G', \Sigma') / P_3'[x, v_3]$, and let H'' be the directed graph obtained from H' by deleting all arcs entering v_3 except for the arc on P_3' and then contracting the arcs in $P_3'[x, v_3]$. Now (G'', Σ'') and H'' certainly satisfies (1), (2) and (4). Moreover, by the hypothesis of this case, it is easy to see that

(G'', Σ'') also satisfies (3). However, this contradicts the minimality of (G', Σ'). This proves 4.8.2.

By possibly changing P_1', P_2', and P_3', we may assume that $\tilde{P}_3 = P_3'[v_3, y]$. Let $(G'', \Sigma'') := (G', \Sigma')\backslash\tilde{P}_3/(P_1'[v_1, y] \cup P_2'[v_1, y])$, $H'' := H'\backslash\tilde{P}_3/(P_1'[v_1, y] \cup P_2'[v_1, y])$, and, for $i = 1, 2$, let $P_i'' := P_i'[x, v_i]$. Note that P_1'', P_2'' and P_3'' are internally vertex disjoint (x, y)-dipaths in H''. Thus (G'', Σ'') and H'' certainly satisfy (1) and (2). Moreover, by 4.8.2, (G'', Σ'') and H'' satisfy (4). By 4.8.1, for $i \in \{1, 2\}$, there exists an odd-circuit cover B of (G'', Σ'') such that $B - (P_i'' \cup \{e\})$ is a transversal of $\mathscr{C}_{\bar{e}}$. By (3), there exists an odd-circuit cover B of (G', Σ') such that $B - (P_3' \cup \{e\})$ is a transversal of $\mathscr{C}_{\bar{e}}$. Hence, $B - \tilde{P}_3$ is an odd-circuit cover of (G', Σ') such that $(B - \tilde{P}_3) - (P_3' \cup \{e\})$ is a transversal of $\mathscr{C}_{\bar{e}}$. We conclude that (G'', Σ'') and H'' satisfy (3). However, this contradicts the minimality of (G', Σ') and H'. ∎

Let (G', Σ'), H' be as given in 4.8, let P_1', P_2', and P_3' be disjoint (x, y)-dipaths in H', and, for $i \in \{1, 2, 3\}$, let $C_i' := P_i' \cup \{e\}$ and let B_i' be a minimal odd-circuit cover of (G', Σ') such that $B_i' - C_i'$ is a transversal of $\mathscr{C}_{\bar{e}}$. We may assume that G' is connected, since otherwise we could delete any component not containing e. As B_1', B_2', and B_3' are minimal odd-circuit covers, they are also signatures of (G', Σ'). Hence, for distinct $i, j \in \{1, 2, 3\}$, $B_i' \Delta B_j'$ is a cut; moreover, $e \notin B_i' \Delta B_j'$. Therefore, for distinct $i, j \in \{1, 2, 3\}$, there exists $U_{ij} \subseteq V(G')$ such that $\delta_{G'}(U_{ij}) = B_i' \Delta B_j'$, and $x, y \notin U_{ij}$. Note that

$$\delta_{G'}(U_{12} \Delta U_{23} \Delta U_{13}) = \delta_{G'}(U_{12}) \Delta \delta_{G'}(U_{23}) \Delta \delta_{G'}(U_{13}) = \emptyset.$$

Moreover, $x, y \notin U_{12} \Delta U_{23} \Delta U_{13}$ and G' is connected. Therefore, $U_{12} \Delta U_{23} \Delta U_{13} = \emptyset$. Let $Y_1 := U_{12} \cap U_{13}$, $Y_2 := U_{12} \cap U_{23}$, $Y_3 := U_{13} \cap U_{23}$, and let $Y_0 := V(G') - (Y_1 \cup Y_2 \cup Y_3)$; thus, $x, y \in Y_0$.

4.9. *For $i \in \{1, 2, 3\}$, C_i' is a circuit in $G'[Y_0 \cup Y_i]$.*

Proof. By symmetry we may assume that $i = 1$. Recall that $x, y \in Y_0$. Moreover, P_1' is disjoint from B_2' and B_3'. Thus, if $f \in P_1' \cap B_1'$, then $f \in \delta_{G'}(U_{12}) \cap \delta_{G'}(U_{13})$. Therefore, either f has one end in Y_0 and the other end in Y_1, or f has an end in Y_2 and an end in Y_3. The edges in $P_1' - B_1'$ are disjoint from each of the cuts $\delta_{G'}(U_{12})$, $\delta_{G'}(U_{13})$, and $\delta_{G'}(U_{23})$. However, P_1' is a path with both ends in Y_0. We conclude that C_1' is contained in $G'[Y_0 \cup Y_1]$, as required. ∎

4.10. *For distinct $i, j \in \{1, 2, 3\}$, there exists a path from $V(P_i')$ to $V(P_j')$ in $G'[U_{ij}]\backslash(B_i' \cup B_j')$.*

Proof. Let $X_i := V(P_i') \cap U_{ij}$ and $X_j := V(P_j') \cap U_{ij}$. We first show that there exists a path from X_i to X_j in $G'[U_{ij}]$. Suppose not, then there exists $X \subseteq U_{ij}$ such that $X_i \subseteq X$, $X \subseteq U_{ij} - X_j$, and $\delta_{G'[U_{ij}]}(X) = \emptyset$. Let $B := B_i' \Delta \delta_{G'}(X)$. Now, B is an odd-circuit cover of (G', Σ'), and $B - \{e\}$ is a transversal of

$\mathscr{C}_{\bar{e}}$, contrary to part (iv) of 4.8. Thus, there exists a path from X_i to X_j in $G'[U_{ij}]$ as claimed.

Note that an edge of $G'[U_{ij}]$ is contained in B'_i if and only if it is also contained in B'_j. Let P be a path from X_i to X_j in $G'[U_{ij}]$ minimizing $|P \cap B'_i|$. Suppose that there exists an edge $f \in P \cap B'_i$. Now, f is in both B'_i and B'_j, so there exists an odd-circuit $C \in \mathscr{C}_{\bar{e}}$ of (G', Σ') containing f. Moreover, by the definition of B'_i, we have $C \cap B'_i = \{f\}$. Therefore, C is a circuit of $G'[U_{ij}]$. We can obtain another path P' from X_i to X_j in $G'[U_{ij}]$ by rerouting P through $C - \{f\}$. However $|P' \cap B'_i| < |P \cap B'_i|$, which contradicts our choice of P. We conclude that P is, in fact, disjoint from B'_i, and, hence, also disjoint from B'_j. ∎

Let $B := B'_1 \Delta B'_2 \Delta B'_3$. Thus, B is a signature for (G', Σ'). Now, for distinct $i, j \in \{1, 2, 3\}$ let P'_{ij} be the path from $V(P'_i)$ to $V(P'_j)$ in $G'[U_{ij}] \backslash (B'_i \cup B'_j)$. Let $S = \{e\} \cup P'_1 \cup P'_2 \cup P'_3 \cup P'_{12} \cup P'_{13} \cup P'_{23}$. Each edge in $S - \{e\}$ is in at most one of the sets B'_1, B'_2, and B'_3. Therefore, the odd edges of $(G', B)[S]$ are e and any edge whose ends are in different parts of (Y_0, Y_1, Y_2, Y_3). Let (G_1, Σ_1) be the signed graph obtained from $(G', B)[S]$ by contracting the edges in $S - B$; thus $\Sigma_1 = E(G_1)$. For distinct $i, j \in \{1, 2, 3\}$, let $P''_{ij} = P'_{ij} \cap B$, and, for $l \in \{0, 1, 2, 3\}$ let Y''_l be the set of vertices of G'' corresponding to Y_i. Now, by 4.1, we see that (G'', Σ'') contains a minor isomorphic to odd-K_5, as required. ∎

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